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ELASTIC MODULI OF A NON-RANDOMLY CRACKED BODY

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Abstract—Calculations on the basis of the self-consistent approximation are used to study the effects of non-random distributions of dry and saturated circular cracks on the effective elastic stiffnesses of a cracked body. Analytic and numerical results are given for two special distributions. In the first, the cracks are assumed randomly distributed in planes parallel to a given plane. In the second instance, the crack normals are randomly distributed in parallel planes. In all cases, the magnitude of the crack induced variations depend upon a crack density parameter ϵ which, for circular cracks of radius a is given by $\epsilon = N\langle a^3\rangle$, where N is the number of cracks per unit volume, and upon an additional saturation parameter.

1. INTRODUCTION

This paper addresses the problem of determining the effective elastic properties of a body permeated by certain non-random distributions of cracks. An isotropic homogeneous body is imagined permeated by a distribution of flat cracks, either wetted by a compressible fluid or dry, which renders the body homogeneous but anisotropic on a scale large compared to the dimensions of the cracks. One wants to know how the elastic compliances and elastic wave speeds depend upon the presence of the cracks. Two special cases are examined. In the first, circular cracks are assumed parallel to each other, and in the second, the normals to circular cracks are assumed randomly distributed in parallel planes.

Specific investigations of the effects on elastic properties of distributions of cracks in solids date from the early and mid 1960s. Initial approaches assumed dilute concentrations of cracks. Bristow[1] determined the effect of a small concentration of dry ribbon-shaped and penny-shaped cracks randomly oriented on the elastic constants of a body. He assessed the defects in elastic potential energy for single cracks by assuming that each crack was not influenced by its neighbors. This allowed him to write down quantitative estimates for first order variations of these material properties. Walsh[2] considered the effect of dilute concentrations of circular cracks, dry or fluid filled, again isotropically distributed.

Dilute concentrations of non-randomly distributed cracks have only recently been considered. The results follow from case of a single crack in an isotropic body. Nur[3] investigated the way in which specific applied stresses can be expected to favor the formation and growth of cracks with preferred orientations and used these results to derive expressions for effective elastic compliances of cracked rocks. Anderson *et al.*[4] have presented a numerical study of the seismic velocities for a solid in which the planes of the circular cracks are all parallel. Griggs *et al.*[5] (circular cracks) and R. J. O'Connell and B. Budiansky [unpublished manuscript, 1977] (elliptical cracks) display analytic expressions for the effective elastic constants as well as for the case of cracks whose normals are constrained to lie in parallel planes.

The case of isotropic crack distributions for large crack concentrations was attacked by Budiansky and O'Connell[6] and the ideas that they presented were important to the present study. They introduced and clarified several important concepts. Firstly, they implemented the self-consistent (SC) scheme to extend their analysis to the domain of large crack concentrations. This technique, developed independently by Budiansky[7] and Hill[8] for composite materials attempts to account for inclusion interactions by estimating the actual behavior of an inclusion in the composite body as that of a single inclusion in the equivalent homogeneous body.

Secondly, Budiansky and O'Connell emphasize that variations in effective constants vary with a crack density parameter ϵ, rather than with the crack porosity. Practically speaking, this

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means that the volume concentration of pore space may not be a useful measure of the effect of cracks on moduli. For elliptic cracks, the crack density is defined as

$$\epsilon = \frac{2N}{\pi}\langle A^2/P\rangle. \tag{1.1}$$

There are N cracks per unit volume and A and P are the area and perimeter of the crack, respectively.

Next, they found that the fluid bulk modulus K_F enters into the expressions for effective moduli only in combination with the matrix bulk modulus K and with the crack aspect ratio α through a saturation parameter $\omega = K_F/\alpha K$. When $\omega < 0.01$, the cracks behave as if empty and if $\omega \gg 10$ they appear saturated by an incompressible fluid.

Lastly, the results they derived for elliptic cracks seem to be quite insensitive to crack planform. This may indicate that their formulas are accurate for cracks with arbitrary convex shapes.

A general formulation of the self-consistent analysis of anisotropic composites recently given by Willis[11] is complementary to the present work.

2. THE SELF-CONSISTENT APPROXIMATION

Two methods have been used for deriving the SC equations which govern the effective elastic constants of composites: one, based on energy considerations[7] and the other[8], involving a direct averaging of the components of stress and strain in the constituent phases of the body. The two methods are entirely equivalent. This latter method, which seems to facilitate the analysis for non-random crack distributions, has been adopted in the present study.

Imagine an initially homogeneous isotropic matrix, of volume V, characterized by Young's modulus and Poisson's ratio, E and ν. This body becomes permeated by non-randomly distributed, fluid-saturated cracks, each of whose linear dimensions is small compared to $V^{1/3}$. The family of admissible crack distributions consists of those which render the body homogeneous but anisotropic in the large; i.e. on a scale large compared with the crack dimensions. It is possible to derive very general expressions governing the effective moduli which apply to bodies permeated by homogeneous distributions of fluid-filled inclusions of arbitrary shape.

Apply some stresses to this body. Let $\bar{\epsilon}_{ij}$, $\bar{\sigma}_{ij}$ be the average strains and stresses experienced by the body, which will be connected by the effective compliances $\bar{M}_{ijkl}$:

$$\bar{\epsilon}_{ij} = \bar{M}_{ijkl}\bar{\sigma}_{kl}. \tag{2.1}$$

The components of the effective tensor $\bar{M}_{ijkl}$ display the usual symmetries in the subscripts, $\bar{M}_{ijkl} = \bar{M}_{ijlk} = \bar{M}_{jikl} = \bar{M}_{klij}$, etc. Equivalently, we may write

$$\bar{\epsilon}_i = \bar{M}_{ij}\bar{\sigma}_j. \tag{2.2}$$

The components of $\bar{\epsilon}_i$, $\bar{\sigma}_i$ are

$$\bar{\epsilon}_x,\ \bar{\epsilon}_y,\ \bar{\epsilon}_z,\ \bar{\gamma}_{yz},\ \bar{\gamma}_{zx},\ \bar{\gamma}_{xy}$$

and

$$\bar{\sigma}_x,\ \bar{\sigma}_y,\ \bar{\sigma}_z,\ \bar{\tau}_{yz},\ \bar{\tau}_{zx},\ \bar{\tau}_{xy},$$

respectively. Here, the γ's are engineering shear strains. The components $\bar{M}_{ij}$ are defined in the appropriate manner to connect the ϵ's and σ's; $\bar{M}_{11} = \bar{M}_{1111}$, $\bar{M}_{12} = \bar{M}_{1122}$, $\bar{M}_{14} = \bar{M}_{1123}$, etc.

The average strain of the body $\bar{\epsilon}_{ij}$ can be written

$$\bar{\epsilon}_{ij} = \frac{1}{V}\int_V \epsilon_{ij}\,\mathrm{d}V = \frac{1}{V}\int_{V_m} \epsilon_{ij}\,\mathrm{d}V + \frac{1}{V}\int_{V_F} \epsilon_{ij}\,\mathrm{d}V. \tag{2.3}$$

Suffix "m" (or no suffix) refers to quantities pertaining to the matrix phase; suffix "F" to those of the fluid inclusion phase. Clearly,

$$V_m + V_F = V,$$

and if

$$\bar{\epsilon}^m_{jk} = \frac{1}{V_m}\int_{V_m} \epsilon_{jk}\,\mathrm{d}V,$$

then

$$\bar{\epsilon}_{ij} = (1-\eta)\bar{\epsilon}^m_{ij} + \eta\bar{\epsilon}^F_{ij} \tag{2.4}$$

where $\eta = V_F/V$ is the volume concentration of the inclusion phase.

Similarly,

$$\bar{\sigma}_{ij} = (1-\eta)\bar{\sigma}^m_{ij} + \eta\bar{\sigma}^F_{ij}. \tag{2.5}$$

A further relationship connects the average stress components $\bar{\sigma}_{ij}$ with the applied tractions $\sigma^\infty_{ij}n_j$: $\bar{\sigma}_{ij} = \sigma^\infty_{ij}$ [8].

The constitutive relations connecting the stresses and strains in the matrix, fluid phase and composite body are

$$\epsilon^m_{ij} = M^m_{ijkl}\sigma^m_{kl} \tag{2.6}$$

$$\sigma^F_{ij} = (K_F\delta_{ij}\delta_{kl})\epsilon^F_{kl} \tag{2.7}$$

$$\bar{\epsilon}_{ij} = \bar{M}_{ijkl}\bar{\sigma}_{kl}. \tag{2.8}$$

Since the matrix is isotropic,

$$M^m_{ijkl} = \frac{1}{E}\left[\frac{1}{2}(1+\nu)(\delta_{ik}\delta_{jl} + \delta_{jk}\delta_{il}) - \nu\delta_{ij}\delta_{kl}\right]. \tag{2.9}$$

To determine the single component $\bar{M}_{ijpq}$, apply a test stress

$$\sigma^\infty_{ij} = \bar{\sigma}_{ij} = \frac{\sigma}{2}(\delta_{ip}\delta_{jq} + \delta_{iq}\delta_{jp}) \tag{2.10}$$

(i.e. only $\bar{\sigma}_{pq} = \bar{\sigma}_{qp} \neq 0$) in (2.8) and focus attention on

$$\bar{\epsilon}_{ij} = \bar{M}_{ijpq}\sigma. \tag{2.11}$$

Here, the fact that $\bar{\sigma}_{ij}n_j$ is the applied traction is used.

Define the quantity $\epsilon^F_{ijpq} \equiv \epsilon^F_{ij}$ to be inclusion strain which arises due to the action of the stress $\bar{\sigma}_{pq}$. Equations (2.4), (2.6) and (2.11) can be combined to yield

$$\bar{M}_{ijpq}\sigma = (1-\eta)M^m_{ijkl}\bar{\sigma}^m_{kl} + \eta\epsilon^F_{ijpq} \tag{2.12}$$

and (2.5) and (2.7) give

$$(1-\eta)\bar{\sigma}^m_{kl} = \bar{\sigma}_{kl} - \eta K_F\delta_{kl}\delta_{rs}\epsilon^F_{rspq}. \tag{2.13}$$

When eqns (2.12) and (2.13) are combined, the final result

$$\bar{M}_{ijpq} = M^m_{ijpq} + \eta(\delta_{ir}\delta_{js} - K_F M^m_{ijkk}\delta_{rs})\left\langle\frac{\epsilon^F_{rspq}}{\sigma}\right\rangle \tag{2.14}$$

is obtained. Here, $\langle \epsilon^F_{rspq} \rangle$ is the average inclusion strain ϵ_{rs} due to the far stress having only $\bar{\sigma}_{pq} = \bar{\sigma}_{qp}$ as its non-zero components.

Equation (2.14) is the basic governing equation for the self-consistent calculation of effective elastic constants of bodies permeated by a homogeneous inclusion phase (not necessarily cracks). Since all quantities on the r.h.s. other than the right-most factor are (in principal) known, the problem has been reduced to determining this average ratio of strain to stress.

The self-consistent approximation is employed as it has been articulated by Budiansky[7] and by Hill[8]. The assumption is made that each individual inclusion sees itself as being a single crack embedded in an otherwise infinite and homogeneous body, but one which is characterized by the as yet unknown constants of the composite body. Thus, the formidable problem of the behavior of a non-randomly distributed ensemble of inclusions in an isotropic body is transformed to the more tractable one of single inclusion behavior in an anisotropic body.

In the remainder of this paper, the foregoing remarks will be applied to the special case of inclusions in the form of cracks, and only elliptical or circular cracks.

There is an apparent difficulty in attempting to apply eqn (2.14) to the special case of cracks; the inclusion strains become singular as the porosity η goes to zero. Simultaneously, the inclusion strains become singular and the aspect ratio vanishes. (The crack can be regarded as the limiting case of an ellipsoid characterized by semi-axes a, b, c, where $a \geq b \geq c$; define the thickness aspect ratio as $\alpha = c/b$.) Nevertheless, the product $\eta\langle \epsilon^F/\sigma \rangle$ remains finite. Indeed, it is the limit

$$\lim_{\alpha \to 0} \left\langle \frac{\epsilon^F_{pq}}{\sigma} \right\rangle$$

which must remain finite, so that (2.14) becomes

$$\bar{M}_{ijpq} = M^m_{ijpq} + (\delta_{ir}\delta_{js} - K_F M^m_{ijkk}\delta_{rs}) \lim_{\alpha \to 0} \left\langle \frac{\alpha \epsilon^F_{rspq}}{\sigma} \right\rangle \lim_{\substack{\eta \to 0 \\ \alpha \to 0}} \left(\frac{\eta}{\alpha} \right).$$

For circular cracks,

$$\lim_{\substack{\eta \to 0 \\ \alpha \to 0}} \eta/\alpha = 4/3\pi\epsilon$$

where $\epsilon = N\langle a^3 \rangle$ is the crack density parameter introduced by Budiansky and O'Connell[6]; N is the number of cracks per unit volume and a the crack radius. Thus, the governing equations for the effective elastic constants permeated by flat circular cracks are

$$\bar{M}_{ijpq} = M^m_{ijpq} + \frac{4}{3}\pi\epsilon(\delta_{ir}\delta_{js} - K_F M^m_{ijkk}\delta_{rs}) \lim_{\alpha \to 0} \left\langle \frac{\alpha \epsilon^F_{rspq}}{\sigma} \right\rangle. \tag{2.14'}$$

Equation (2.14′) is the basic result of this section and will frequently be referred to below. For dry cracks ($K_F = 0$), eqn (2.14′) becomes

$$\bar{M}_{ijpq} = M^m_{ijpq} + \frac{4}{3}\pi\epsilon \lim_{\alpha \to 0} \left\langle \frac{\alpha \epsilon^I_{ijpq}}{\sigma} \right\rangle. \tag{2.14''}$$

(It seems more appropriate to use the superscript I = inclusion rather than F when dealing with dry cracks.)

The main result of this section, eqn (2.14″), can be used in conjunction with knowledge of a single flat crack in a (suitably defined) anisotropic body to specify the effective elastic moduli of non-randomly cracked bodies. This analysis is presented in Refs. [9, 10]. Some results from this investigation will be here summarized.

Under the action of uniform far stresses, an arbitrary ellipsoidal cavity characterized by semi-axes a, b, c, $a \geq b > c$, deforms into another ellipsoid. Thus, the crack face displacements will also be ellipsoidal, i.e. there exist dimensionless quantities β_i such that the cavity displacements u_i are

$$u_i = \beta_i a \sqrt{\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)}. \tag{2.15}$$

(The explicit recipes for the β_i are detailed in [9, 10].) The strain associated with the crack in the limit as the thickness aspect ratio $\alpha = c/b$ vanishes are related to the cavity displacement magnitudes by

$$\frac{E}{\sigma} \lim_{\alpha \to 0} [\alpha \epsilon_{ij}^I] = \frac{1}{2\sqrt{\gamma}} \beta_p (\delta_{ip}\delta_{j3} + \delta_{jp}\delta_{i3}) \tag{2.16}$$

where $\gamma = b/a$, and superscript "I" refers to field quantities associated with the inclusion. Here, E is some characteristic modulus of the solid, and σ is some characteristic value of the applied stress. Associated with the β_i are a set of influence coefficients C_{ij} relating the crack displacements to the applied stresses σ_{ij}^∞:

$$\sigma_{3j}^\infty = \sigma C_{jk} \beta_k. \tag{2.17}$$

The quantities C_{ij} are symmetric in their subscripts. Stress components other than σ_{3k}^∞ have no influence on the β_i for flat cracks.

3. EFFECTIVE MODULI (DRY CRACKS)

The results of the last section, eqns (2.14″), can be used in conjunction with the behavior of dry cracks in anisotropic media[9, 10] to specify the effective moduli of dry cracked bodies. The general method to carry out this task numerically consists of the following steps.

(1) Assume values of $\bar{M}_{ij}$. The $\bar{M}_{ij}$ are compliances which connect the 6-dimensional stress and strain vectors: $\boldsymbol{\epsilon} = (\epsilon_1, \epsilon_2, \epsilon_3, \gamma_{23}, \gamma_{31}, \gamma_{12})$, $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_{12})$; $\epsilon_i = \bar{M}_{ij}\sigma_j$.

(2) Use these to determine the matrices C_{ij} associated with a single crack. In general, this must be done with computer assistance. Since the average of the limits $\langle \alpha \epsilon_{ij}^I \rangle$ are needed, this computation may have to be carried out many times corresponding to cracks of different orientations with respect to the material frame of reference. The average strain limits $\lim_{\alpha \to 0} \alpha \epsilon_{ijpq}^I$ are now calculated, for a variety of test stresses, by means of these matrices (C_{ij}).

(3) These outputs are combined with the governing SC equations (2.14″) to yield an explicit set of relations

$$\bar{M}_{ij} = \bar{M}_{ij} (\langle \lim_{\alpha \to 0} \alpha \epsilon_{kl}^I \rangle).$$

Using a standard iterative scheme (successive approximation, Newton–Raphson, etc.), the computed average limits of Step 2 are used to refine the initial guesses for $\bar{M}_{ij}$.

(4) Return to Step 1 and proceed until the $\bar{M}_{ij}$ converge to some prescribed tolerance.

Two special examples will be worked through. The first is that of planar transverse isotropy (PTI), in which the cracks are randomly distributed in planes parallel to a given plane. In the second, that of cylindrical transverse isotropy (CTI), the cracks are distributed subject to the constraint that their normals lie randomly in planes parallel to a given plane. In both cases, the cracked solid appears macroscopically transversely isotropic, with the given plane being parallel to the plane of isotropy.

The following derivations will require dealing with quantities referred to two coordinate systems. The first system, to be termed the *crack coordinate system*, is one which is defined with respect to a single crack. The crack is located in the x–y plane of this system whose coordinates are $(x_1, x_2, x_3) = (x, y, z)$. The *material coordinate* system, the second, with coordinates $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (\bar{x}, \bar{y}, \bar{z})$, is one for which the plane of isotropy of the cracked body is the

$\bar{x}$–$\bar{y}$ plane. If a point P has coordinates (x_1, x_2, x_3) in the crack system, then its coordinates $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in the material system are given by means of the tensor transformation matrix (l_{ij}):

$$\bar{x}_i = l_{ij}x_j$$
$$x_i = l_{ji}\bar{x}_j. \tag{3.1}$$

One important simplifying feature of the PTI and CTI analyses to be performed is that the matrix (C_{ij}) (relating crack displacement to applied stress referred to the crack system) has the same representation in both the crack and body coordinate systems. Consequently, only one computation of the matrix (C_{ij}) will need to be made when determining the $\bar{M}_{ij}$: the suitable averaged values can be related to this single calculation.

The remainder of this section will be devoted to carrying out each analysis to the point where computer assistance is called for. For PTI, explicit analytic expressions for the effective constants as functions of crack density can be obtained. For CTI, this point is reached after deriving the expressions

$$\bar{M}_{ij} = \bar{M}_{ij}(\langle \lim_{\alpha\to 0} \alpha\epsilon^I_{kl}\rangle).$$

PTI

The only two effective constants to be altered by the cracks are $\bar{M}_{33} = 1/\bar{E}$ and $\bar{M}_{44} = \bar{M}_{55} = 1/\bar{G}$. With respect to the body axes, the resulting constitutive matrix becomes

$$\bar{M} = \frac{1}{E}\begin{pmatrix} 1 & -\nu & -\nu & & & \\ -\nu & 1 & -\nu & & & \\ -\nu & -\nu & \frac{1}{H} & & & \\ & & & \frac{2(1+\nu)}{\Gamma} & 0 & 0 \\ & & & 0 & \frac{2(1+\nu)}{\Gamma} & 0 \\ & & & 0 & 0 & 2(1+\nu) \end{pmatrix} \tag{3.2}$$

where $H = \bar{E}/E$ and $\Gamma = \bar{G}/G = 2(1+\nu)\bar{G}/E$.

One uses special cases of (2.14″) to derive H and Γ. For H and Γ,

$$\frac{1}{\bar{E}} = \frac{1}{E} + \frac{4}{3}\pi\epsilon\left\langle \lim_{\alpha\to 0} \frac{\alpha\bar{\epsilon}^I_{3333}}{\sigma}\right\rangle \tag{3.3}$$

$$\frac{1}{\bar{G}} = \frac{1}{G} + \frac{4}{3}\pi\epsilon\left\langle \lim_{\alpha\to 0} \frac{\alpha\bar{\gamma}^I_{2323}}{\tau}\right\rangle \tag{3.4}$$

where test stresses $\sigma^\infty_{33} = \sigma$, $\tau^\infty_{23} = \tau$ have respectively and separately been applied.

Since the cracks are assumed parallel to the $\bar{x}$–$\bar{y}$ plane, it is possible, and highly desirable, to choose a crack system x_i such that each axis 0–x_i is parallel to the 0–$\bar{x}_i$ axis so that $l_{ij} = \delta_{ij}$. With this choice of crack system, the limit inclusion strains of eqns (3.3) and (3.4) are given precisely by the crack displacement magnitudes β_i:

$$\lim_{\alpha\to 0}\left\langle \frac{\alpha\bar{\epsilon}^I_{33}}{\sigma}\right\rangle = \beta_3$$

$$\lim_{\alpha\to 0}\left\langle \frac{\alpha\bar{\gamma}^I_{23}}{\sigma}\right\rangle = \beta_1$$

where the β's can be obtained from eqns (2.30) and (2.6) of Ref. [10]. Substituting these expressions into (3.3) and (3.4) yields the desired equations for H and Γ.

$$\frac{1}{H}=1+\frac{8}{3}\sqrt{\left[2\left(\frac{1}{H}-\nu^2\right)\right]}\sqrt{\left[(1+\nu)\left(\frac{1}{\Gamma}-\nu\right)+\sqrt{\left[(1-\nu^2)\left(\frac{1}{H}-\nu^2\right)\right]}\right]}\epsilon \tag{3.5}$$

$$\frac{1}{\Gamma}=1+\frac{8}{3}\sqrt{\left[2\left(\frac{1-\nu}{1+\nu}\right)\right]}\sqrt{\left[(1+\nu)\left(\frac{1}{\Gamma}-\nu\right)+\sqrt{\left[(1-\nu^2)\left(\frac{1}{H}-\nu^2\right)\right]}\right]}$$
$$\times\left\{1+\sqrt{\Gamma}\sqrt{\left[\frac{1}{2}\left(\frac{1-\nu}{1+\nu}\right)\right]}\sqrt{\left[(1+\nu)\left(\frac{1}{\Gamma}-\nu\right)+\sqrt{\left[(1-\nu^2)\left(\frac{1}{H}-\nu^2\right)\right]}\right]}\right\}^{-1}\epsilon. \tag{3.6}$$

Solutions to these equations are graphed in Fig. 1 for $\nu=1/4$. These curves visually demonstrate what analysis of eqns (3.5) and (3.6) bear out: There is no finite critical crack density where either $\bar{E}$ or $\bar{G}$ vanish. As there is very limited scope for sufficiently severe crack intersection among purely parallel cracks, this result is entirely expected.

CTI

Three independent elastic constants are affected by the presence of cracks. These are $\bar{M}_{44}=\bar{M}_{55}=1/\bar{G}$, $\bar{M}_{66}=1/G^*$ and $\bar{M}_{11}=\bar{M}_{22}=1/\bar{E}$. The constitutive matrix becomes

$$\bar{\mathbf{M}}=\frac{1}{E}\begin{pmatrix} \frac{1}{H} & -\bar{\nu} & -\nu & & & \\ -\bar{\nu} & \frac{1}{H} & -\nu & & & \\ -\nu & -\nu & 1 & & & \\ & & & \frac{2(1+\nu)}{\Gamma} & 0 & 0 \\ & & & 0 & \frac{2(1+\nu)}{\Gamma} & 0 \\ & & & 0 & 0 & \frac{2(1+\nu)}{\Gamma^*} \end{pmatrix} \tag{3.7}$$

where, for convenience, the quantities $\Gamma=\bar{G}/G$, $\Gamma^*=G^*/G$, $H=\bar{E}/E$, and

$$\bar{\nu}=\frac{1+\nu}{\Gamma^*}-\frac{1}{H} \tag{3.8}$$

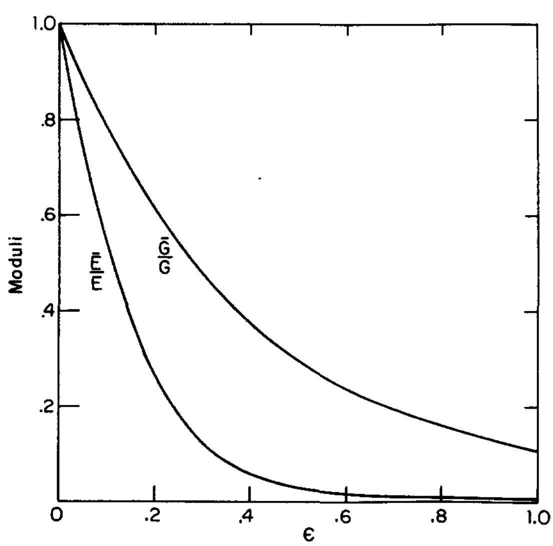

Fig. 1. PTI elastic constants; dry, circular cracks.

have been defined. An additional derived quantity,

$$\nu^* = H\bar{\nu} \tag{3.9}$$

represents the ratio of fractional lateral contraction $\bar{\epsilon}_{11}$ to the linear strain $\bar{\epsilon}_{22}$ when a test stress $\bar{\sigma}_{11}$ is applied to the cracked body. The constants of (3.7) are referred to the material coordinate system.

The eqn (2.14″), when specialized to CTI, will take the following form. When a test stress $\bar{\sigma}_{11}^{\infty} = \sigma$ is applied, then

$$\frac{1}{\bar{E}} = \frac{1}{E} + \frac{4}{3}\pi\epsilon\left\langle \lim_{\alpha\to 0}\frac{\alpha\bar{\epsilon}_{1111}^{I}}{\sigma}\right\rangle. \tag{3.10}$$

If a test shear $\bar{\tau}_{23}^{\infty} = \tau$ is applied,

$$\frac{1}{\bar{G}} = \frac{1}{G} + \frac{4}{3}\pi\epsilon\left\langle \lim_{\alpha\to 0}\frac{\alpha\bar{\gamma}_{2323}^{I}}{\tau}\right\rangle. \tag{3.11}$$

As a third test stress, it is convenient to apply hydrostatic pressure $\bar{\sigma}_{ij}^{\infty} = p\delta_{ij}$ in order to determine an effective bulk modulus $\bar{K}$. A modest generalization of (2.14″) leads to

$$\frac{1}{\bar{K}} = \frac{1}{K} + \frac{4}{3}\pi\epsilon\left\langle \lim_{\alpha\to 0}\frac{\alpha\bar{\epsilon}_{jj}^{I}}{p}\right\rangle. \tag{3.12}$$

If $\bar{K} = \sigma/\bar{\epsilon}_{kk}$, then

$$\frac{1}{\bar{K}} = \frac{1}{E}\left[1 + \frac{2}{H} - 2\bar{\nu} - 4\nu\right]. \tag{3.13}$$

The modulus G^* is obtained by means of the auxiliary relation

$$\frac{1}{G^*} = \frac{1}{E}\left(1 + \frac{4}{H} - 4\nu\right) - \frac{1}{\bar{K}}. \tag{3.14}$$

Contemplate a single crack within a body characterized by the constitutive matrix (3.7) whose plane is inclined at an angle θ with the x–z plane of the body system. The matrix (l_{ij}) has the following matrix representation:

$$\mathbf{l} = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ -\sin\theta & 0 & \cos\theta \\ 0 & 1 & 0 \end{pmatrix} \tag{3.15}$$

and note that $\mathbf{l}^{-1} = \mathbf{l}^T$. For a series of test stresses $\bar{\sigma}_{rs}^{\infty}$ it is necessary to get the resulting crack displacements as a function of θ. These are then averaged over all possible values of θ and substituted into (3.10)–(3.12) to specify equations $\bar{M}_{ij}(\langle \lim_{\alpha\to 0} \alpha\bar{\epsilon}_{kl}^{I}\rangle)$.

A state of stress $\bar{\sigma}_{rs}^{\infty}$ applied to the body far from the crack gives rise to limiting crack strains. When referred to the body frame of reference, these are given by

$$\frac{E}{\sigma}\lim_{\alpha\to 0}\alpha\bar{\epsilon}_{ij}^{I} = \frac{E}{\sigma}\lim_{\alpha\to 0}[\alpha l_{ip}l_{jq}\epsilon_{pq}^{I}] = l_{ip}l_{jq}\frac{E}{\sigma}\lim_{\alpha\to 0}[\alpha\epsilon_{pq}^{I}]. \tag{3.16}$$

Now, the equations which relate crack strain singularities with displacements and displacements with applied stress are

$$\frac{E}{\sigma}\lim_{\alpha\to 0}(\alpha\epsilon_{ij}^{I}) = \frac{\beta_p}{2}a_{pij} \tag{3.17}$$

where

$$a_{pij} = \delta_{ip}\delta_{j3} + \delta_{jp}\delta_{i3} \tag{3.18}$$

and

$$\beta_p = \frac{1}{\sigma} C^{-1}_{pk}\sigma^{\infty}_{3k}. \tag{3.19}$$

The end result of melding eqns (3.16)–(3.19) is

$$E \lim_{\alpha \to 0} [\alpha \bar{\epsilon}^I_{ij}] = \frac{1}{2} l_{ip} l_{jq} C^{-1}_{km} l_{sm} l_{r3} \bar{\sigma}^{\infty}_{rs} a_{kpq} = \frac{1}{2} (l_{ik} l_{j3} + l_{i3} l_{jk}) C^{-1}_{km} l_{sm} l_{rs} \bar{\sigma}^{\infty}_{rs}. \tag{3.20}$$

The term C^{-1}_{ij} is independent of the angle θ and furthermore is a diagonal tensor:

$$C^{-1}_{ij} = d_{(i)}\delta_{ij} \tag{3.21}$$

where the parentheses about a subscript suspend the summation convention. Substituting (3.21) into (3.20) yields the final relation

$$E \lim_{\alpha \to 0} [\alpha \bar{\epsilon}^I_{ij}] = \frac{d_{(k)}}{2} l_{sk} l_{r3} \sigma^{\infty}_{rs} [l_{ik} l_{j3} + l_{i3} l_{jk}]. \tag{3.22}$$

To determine H, apply a test stress $\bar{\sigma}^{\infty}_{11} = \sigma$ and note, from (3.22), that

$$\lim_{\alpha \to 0} \left[\frac{\alpha \epsilon_{11}}{\sigma}\right] = \frac{\sin^2\theta}{E} (\cos^2\theta d_1 + \sin^2\theta d_3). \tag{3.23}$$

Since $\langle \cos^4\theta \rangle = \langle \sin^4\theta \rangle = 3/8$ and $\langle \cos^2\theta \sin^2\theta \rangle = 1/8$, (3.23) can be substituted into (3.10) to yield

$$\frac{1}{H} = 1 + \frac{\pi\epsilon}{6}(d_1 + 3d_3) \quad \text{(CTI, dry)}. \tag{3.24}$$

Next, to determine Γ, apply a test stress $\tau^{\infty}_{23} = \tau$. From (3.22)

$$\lim_{\alpha \to 0} \left[\frac{\alpha \bar{\gamma}^I_{23}}{\tau}\right] = \frac{\cos^2\theta}{E} d_2 \tag{3.25}$$

so that

$$\frac{1}{\Gamma} = 1 + \frac{\pi\epsilon}{3} \frac{d_2}{(1+\nu)} \quad \text{(CTI, dry)}. \tag{3.26}$$

Here, the fact $\langle \cos^2\theta \rangle = 1/2$ was used.

When hydrostatic pressure is applied far from the crack, the limit crack strain is independent of θ (owing to the isotropic nature of the pressure tensor $\bar{\sigma}^{\infty}_{ij} = p\delta_{ij}$) and also

$$\lim_{\alpha \to 0} \left[\frac{\alpha \bar{\epsilon}^I_{KK}}{p}\right] = \frac{d_3}{E}. \tag{3.27}$$

From (3.12), this implies

$$\frac{K}{\bar{K}} = 1 + \frac{4}{9}\pi\epsilon \frac{d_3}{(1-2\nu)} \tag{3.28}$$

so that, in conjunction with (3.14)

$$\frac{1}{\Gamma^*} = 1 + \frac{\pi\epsilon}{3}\frac{(d_1 + d_3)}{(1+\nu)} \quad \text{(CTI, dry)}. \tag{3.29}$$

Also, from (3.8),

$$\bar{\nu} - \nu = \frac{\pi\epsilon}{6}(d_1 - d_3) \quad \text{(CTI, dry)}. \tag{3.30}$$

Thus, the dry cracked CTI body has effective constants specified by eqns (3.24), (3.26) and (3.29) or (3.30). The terms d_i, being independent of θ, are unambiguously defined as being the diagonal elements of the matrix C_{ij}^{-1} (see eqn 3.21) associated with the crack.

These effective stiffnesses have been obtained for $\nu = 0.25$ and are shown in Fig. 2. Note that above a crack density of $\epsilon \approx 0.65$, ν^* is less than zero. Furthermore, despite the great opportunities for crack intersection afforded by the cylindrical distributions, these graphs imply that there is no finite critical crack density for which $\bar{E}$, $\bar{G}$ or $\bar{G}^*$ vanish.

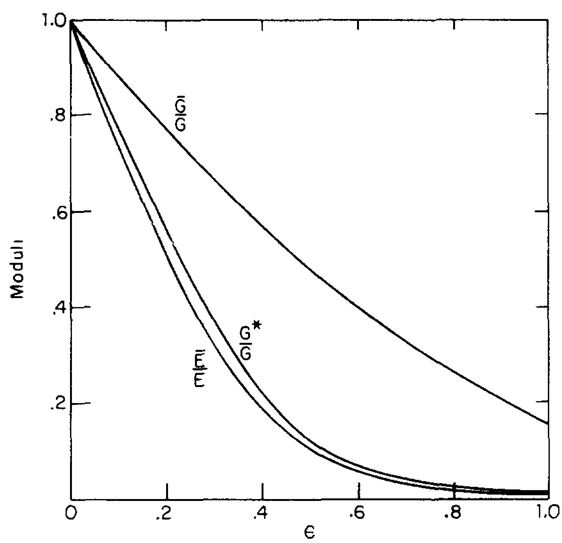

Fig. 2. CTI elastic constants; dry, circular cracks.

4. EFFECTIVE MODULI (SATURATED CRACKS)

In this section, the effect of cracks wetted by a fluid of bulk modulus K_F on effective moduli will be determined. The two examples of PTI and CTI configurations of circular cracks will be discussed in detail.

The following analysis is subject to two assumptions. Firstly, the fluid in each crack is isolated from that of its neighbors. The resulting moduli, then, will be appropriate to stress changes that occur with sufficient rapidity to prevent communication of fluid pressure between cracks, corresponding to the case of sufficiently high frequency elastic waves. A second assumption is that $K_F/K \ll 1$. (For air-filled dry cracks, $K_F/K \sim 10^{-6}$. For cool water-filled cracks, $K_F/K \sim 0.03$ [6].)

In dealing with saturated cracks, the assumption of zero crack volume cannot be involved. A small non-zero crack volume v_c must be assumed, and the saturation parameter

$$\omega = \frac{8}{3}\frac{A^2}{Pv_c}\frac{K_F}{K} \tag{4.1}$$

will then enter the analysis in an essential way. A and P are, respectively, the area and perimeter of the cracks. For circular cracks, this expression reduces to $\omega = K_F/\alpha K$, where α is the thickness aspect ratio. The empty crack case is recovered for $\omega = 0$.

The following procedure will be followed. As in all SC analysis, the elastic behavior of a single fluid-filled crack in a suitably defined elastic matrix must be resolved. The strains and displacements for this crack can be straightforwardly obtained from those of a single dry crack in the same body.

The theorem that relates the behavior of dry and fluid-filled cracks follows trivially from the work of Eshelby[12]. It has the following statement.

Let σ_{jk}^{∞} be uniform stresses in the inclusion frame applied far away from the inclusion. (These stresses give rise to uniform inclusion stresses and strains σ_{jk}^{I}, ϵ_{jk}^{I}.) Then there exist two tensors λ_{jklm}, μ_{jklm} depending *only* upon inclusion geometry and matrix moduli and *not* upon inclusion moduli such that

$$E\lambda_{jklm}\epsilon_{lm}^{I} + \mu_{jklm}\sigma_{lm}^{I} = \sigma_{jk}^{\infty}. \tag{4.2}$$

Here again, E is some one of the matrix moduli.

The proof of this theorem, which relies upon the overall linearity of the problem and upon the uniformity of the inclusion strains, will not be given here.

PTI

As in the previous section, the solid will be characterized by the constitutive matrix (eqn 3.2). Since both voids and fluids cannot resist shear, the procedure by which eqn (3.6) relating Γ to H was derived is independent of the presence or absence of crack fluid. It will thus continue to be valid for saturated cracks.

To determine a second relation between H and Γ, it is most convenient to apply hydrostatic pressure $\bar{\sigma}_{ij}^{\infty} = \sigma_{ij}^{\infty} = p\delta_{ij}$ far from the body. (Crack and body frame coordinate axes are chosen to coincide.) The bulk modulus is given by

$$\frac{1}{\bar{K}} = \frac{\epsilon_{kk}}{\bar{p}} = \frac{1}{E}\left(2 + \frac{1}{H} - 6\nu\right) \tag{4.3}$$

so that H is known whenever $E/\bar{K}$ is. The constant $\bar{K}$ will be determined from

$$\frac{1}{\bar{K}} = \frac{1}{K} + \frac{4}{3}\pi\alpha\epsilon\left\langle\frac{\epsilon_{kk}^{F}}{p}\right\rangle. \tag{4.4}$$

Since the crack volume is finite, it is quite proper to work with crack strains when dealing with fluid-filled cracks.

From (4.2), there will exist some constants $\lambda \equiv \lambda_{jjkk}$, $\mu \equiv \mu_{jjkk}$ such that

$$\bar{K}\lambda\epsilon_{kk}^{F} + \mu p^{F} = p. \tag{4.5}$$

Consider two special values of K^F in order to evaluate λ and μ.

When $K^F = \bar{K}$, then $p^F = p$, $\epsilon_{kk}^{F} = p/\bar{K}$. Therefore,

$$\lambda + \mu = 1. \tag{4.6}$$

Since the λ, μ are *independent* of K^F, the result (4.6) is valid for arbitrary K^F.

Next, suppose the crack is dry, $K^F = 0$. Then $p^F = 0$ and

$$\lambda = \frac{1}{\bar{K}(\epsilon_{kk}^{I}/p)_0}. \tag{4.7}$$

(The subscript "0" identifies quantities associated with the empty crack.) The strain $(\epsilon_{kk}^{I})_0 = p/E\beta_3/\alpha + 0(p/\bar{K})$ is known from the dry crack analysis of the preceding sections, where β_3 is the appropriate crack face displacement obtained from (2.30) and (2.6) of Ref. [10]:

$$\beta_3 = \frac{2}{\pi}\sqrt{\left[2\left(\frac{1}{H}-\nu^2\right)\right]}\sqrt{\left[(1+\nu)\left(\frac{1}{\Gamma}-\nu\right)+\sqrt{\left[(1-\nu^2)\left(\frac{1}{H}-\nu^2\right)\right]}\right]}. \tag{4.8}$$

Equations (4.6) and (4.7) determine λ and μ uniquely.

For arbitrary K^F, use the fact that $p^F = K_F\epsilon^F_{kk}$, together with (4.6) and (4.7) plus the assumption $K_F/K \ll 1$ to determine that

$$\left\langle\frac{p}{p^F}\right\rangle = \frac{p}{p^F} = 1 + \frac{3(1-2\nu)}{\omega\beta_3} \tag{4.9}$$

so that, together with (4.4),

$$\frac{K}{\bar{K}} = 1 + \frac{4\pi\epsilon}{9(1-2\nu)}\cdot\frac{\beta_3}{\left[1+\dfrac{\omega\beta_3}{3(1-2\nu)}\right]}$$

where $\omega = K_F/\alpha K$. Let

$$D = \left[1+\frac{\omega\beta_3}{3(1-2\nu)}\right]^{-1}. \tag{4.10}$$

In combination with (4.3) and (4.8),

$$\frac{1}{H} = 1 + \frac{16}{3}\sqrt{\left[\frac{(1/H)-\nu^2}{2}\right]}\sqrt{\left[(1+\nu)\left(\frac{1}{\Gamma}-\nu\right)+\sqrt{\left[(1-\nu^2)\left(\frac{1}{H}-\nu^2\right)\right]}\right]}D\epsilon.$$

(PTI Saturated) (4.11)

Equation (4.11) is very general and encompasses both dry and saturated cracks. The case $D = 1$ corresponds to dry cracks, while $D = 0$ corresponds to crack saturation by a hard fluid.

Equations (4.11) and (3.6) fully describe the saturated PTI cracked body. The solution of these relations for $\bar{E}/E$ and $\bar{G}/G$ are graphed in Fig. 3 for $\nu = 0.25$ and for various values of ω. The cracks may be considered dry if $\omega < 0.1$ and wetted by a hard fluid ($\omega = \infty$) when $\omega > 100$.

CTI

The cracked body has a constitutive matrix given by (3.7). The effective constants are specified by eqns (3.10)–(3.12), although it is no longer permissible to take limit of vanishing aspect ratio α in these equations. The same single crack as in the dry crack analysis is examined. The crack strain is given by eqn (3.20):

$$E\alpha\bar{\epsilon}^F_{ij} = \frac{1}{2}(l_{ik}l_{j3}+l_{i3}l_{jk})C^{-1}_{km}l_{sm}l_{rs}\sigma^\infty_{rs}$$

(where a test stress σ^∞_{rs} is applied) but the components C^{-1}_{ij} are, in general, no longer described by the theory of Section 2, since the presence of fluid in the crack will alter the relationship of applied stress to crack displacement.

The tensor C^{-1}_{ij} (or C_{ij}) remains diagonal for saturated cracks. The components C^{-1}_{11} and C^{-1}_{22} relate the stress components σ^∞_{31}, σ^∞_{32} (the far stress is here referred to the crack coordinate system) to the shear crack displacements β_1, β_2. These shear displacements will be unaffected by the presence of fluid, and so the forms for C_{11}, C_{22} will be the same as in the dry crack analysis. Only C_{33} will be different.

To discuss the saturated component $C^{-1}_{33} = d_3{}^F$, imagine the application of hydrostatic pressure at infinity: $\sigma^\infty_{ij} = p\delta_{ij}$. The Eshelby theorem (4.2) is invoked in the same way as for PTI,

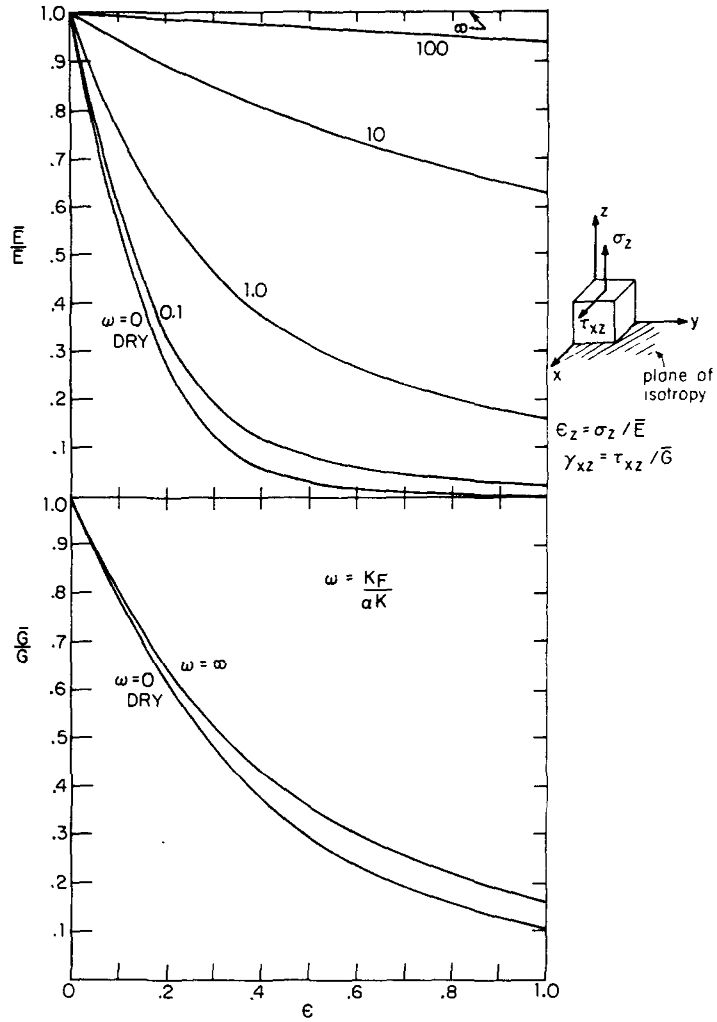

Fig. 3. PTI elastic constants; dry and saturated circular cracks.

so that certain constants λ, μ are defined by eqn (4.5). Carrying out an identical procedure, by which two special cases, that of $K_F = \bar{K}$ and $K_F = 0$, are used to solve for these two parameters, results in the following formal expression for p^F/p:

$$\left\langle \frac{p}{p^F} \right\rangle = 1 + \frac{3(1-2\nu)}{\omega d_3} \tag{4.12}$$

where the notation $d_3 = (C_{33}^{-1})_0$ has been used and refers to the tensor component computed for dry cracks. Equation (4.12) with (3.12) imply that

$$\frac{K}{\bar{K}} = 1 + \frac{4}{9}\pi\epsilon \frac{Dd_3}{(1-2\nu)} \tag{4.13}$$

where

$$D = \left[1 + \frac{d_3\omega}{3(1-2\nu)}\right]^{-1}. \tag{4.14}$$

Comparison of (4.13) with (3.28) suggests that the saturated displacement is simply obtained from the dry calculation by means of the factor D:

$$d_3^F = Dd_3. \tag{4.15}$$

Thus, the equations for the effective moduli are obtained from the dry relations, (3.24), (3.26), (3.29), (3.30) by replacing d_3 in those equations by (Dd_3):

$$\frac{1}{H} = 1 + \frac{\pi\epsilon}{6}[d_1 + 3Dd_3] \tag{4.16}$$

$$\frac{1}{\Gamma} = 1 + \frac{\pi\epsilon}{3}\frac{d_2}{(1+\nu)} \tag{4.17}$$

$$\frac{1}{\Gamma^*} = 1 + \frac{\pi\epsilon}{3}\frac{(d_1 + Dd_3)}{(1+\nu)} \quad \text{CTI, saturated} \tag{4.18}$$

$$\bar{\nu} - \nu = \frac{\pi\epsilon}{6}(d_1 - Dd_3). \tag{4.19}$$

Again, the adoption of this D-notation allows consolidation of the description of dry and saturated CTI bodies into one set of equations, (4.16)–(4.19). Dry cracked bodies are recovered for $D = 1$.

These solutions are graphed in Fig. 4, which show the moduli for $\nu = 1/4$ and for various values of ω. The dry case is very well approximated by $\omega < 0.1$, and the case $\omega = \infty$ may be used with little error whenever $\omega > 10$. Note, too, the non-uniform variation of ν^* with ω. Demanding that the strain energy of the cracked body associated with a tensile stress σ_{11} remain positive definite leads to the inequality

$$\nu^* < 1 - \nu H.$$

As ϵ increases, $H \to 0$, so that, in the limit of large crack densities, the least upper bound of ν^* is 1.

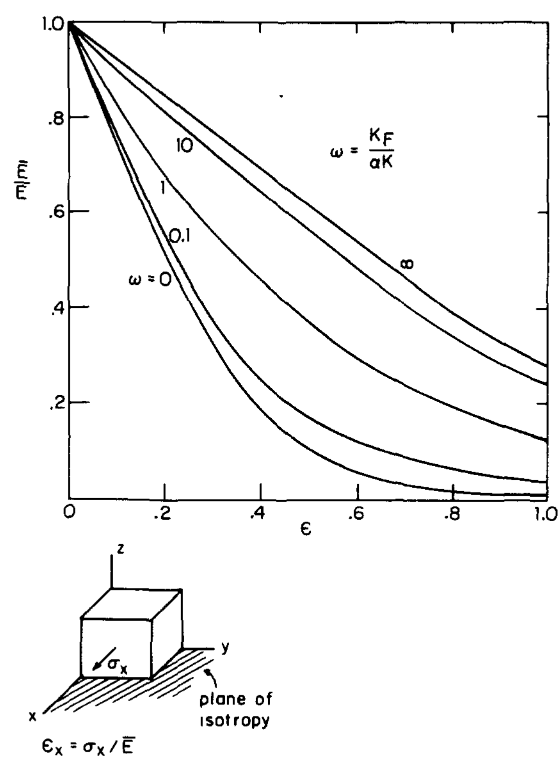

Fig. 4(a). CTI elastic constants; dry and saturated circular cracks, $\bar{E}$.

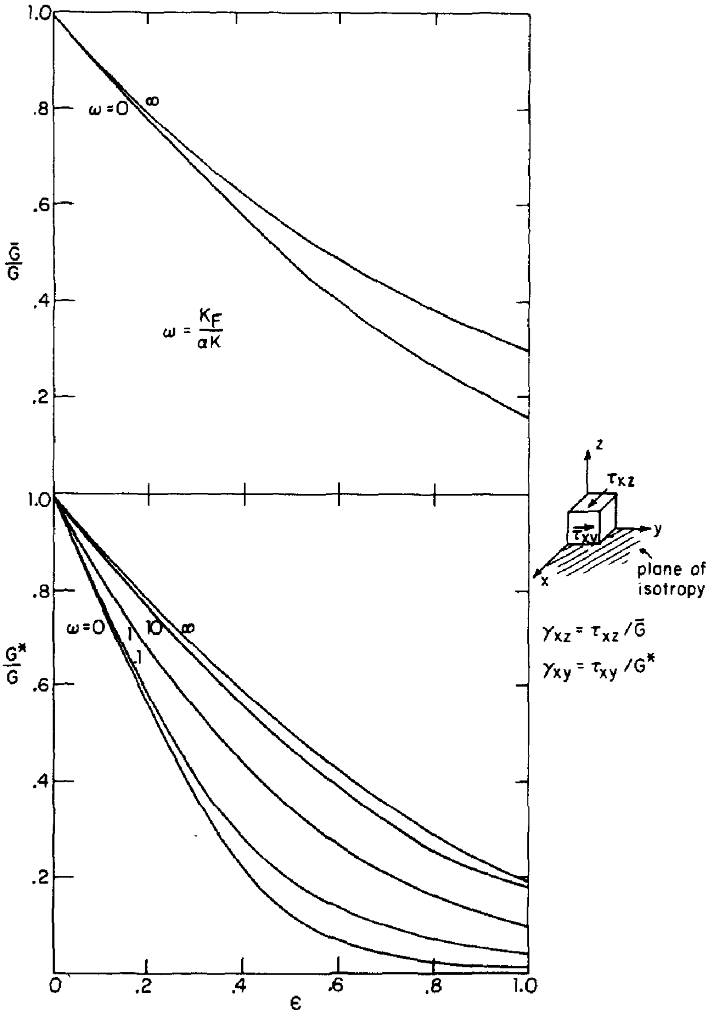

Fig. 4(b, c). CTI elastic constants; dry and saturated circular cracks, $\bar{G}$, G^*.

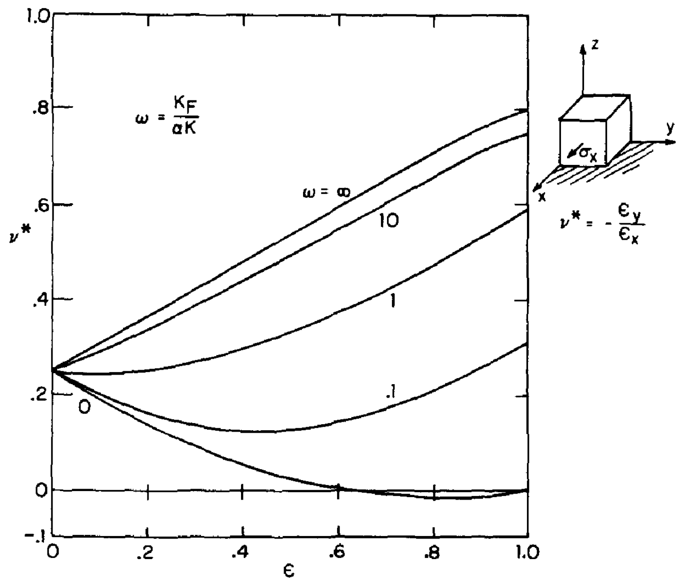

Fig. 4(d). CTI elastic constants; dry and saturated circular cracks, ν^*.

Partial saturation

From eqn (2.14) as a starting point, it is easy to generalize to the case of partial saturation in which only a fraction ξ of the cracks are saturated. One simply replaces the parameter D by $(1-\xi+\xi D)$ in eqn (3.6) and (4.11) for PTI and in (4.16)–(4.19) for CTI.

Seismic velocities

An important geophysical application of effective constant theory is toward the prediction of effective seismic velocities of cracked bodies. In terms of a matrix $\bar{L}_{ij} = \bar{M}_{ij}^{-1}$, we define the quantities

$$\begin{aligned} a &= L_{11} - L_{44} \\ C &= L_{11} - L_{12} - 2L_{44} \\ d &= L_{13} + L_{44} \\ h &= L_{33} - L_{44} \\ H &= \rho v^2 - L_{44} \end{aligned} \tag{4.21}$$

following, e.g. Anderson, Minster and Cole[4]. Here, ρ is the density of the body and v is an elastic wave velocity. The velocity equation in any plane containing the unique axis of a transversely isotropic body is

$$\left(H - \frac{m^2C}{2}\right)[(H - m^2a)(H - n^2h) - m^2n^2d^2] = 0. \tag{4.22}$$

The direction cosines m and n denote the direction of propagation: $m = \sqrt{[1-n^2]}$. Propagation in the direction of the unique axis is denoted by $n = 1$. The wave associated with the root $H = m^2C/2$ is purely transverse. The additional two roots are purely transverse or purely longitudinal only in the directions $n = 0$ or $n = 1$. Parallel to the plane of isotropy, the three velocities of propagation are $(L_{11}/\rho)^{1/2}$, $(L_{44}/\rho)^{1/2}$ and $[(L_{11}-L_{12})/2\rho]^{1/2}$ and correspond to a pressure wave and to two transverse waves, respectively. In the direction of the unique axis the compressional velocity is $(L_{33}/\rho)^{1/2}$ and the two shear wave velocities coincide: $(L_{44}/\rho)^{1/2}$.

This brief recapitulation has been cast in terms of the notation and language developed by geophysicists. It is convenient to introduce a more descriptive notation. To this end, let

$$\begin{aligned} v_{p_\parallel} &= (L_{33}/\rho)^{1/2} \\ v_{p_\perp} &= (L_{11}/\rho)^{1/2} \end{aligned} \tag{4.23}$$

where the symbols $\parallel$, $\perp$ refer to propagation parallel and perpendicular to the unique axis, respectively. Similarly

$$v_{s_{1,\parallel}} = v_{s_{2,\parallel}} \equiv V_{s,\parallel} = (L_{44}/\rho)^{1/2} \tag{4.24}$$

and

$$\begin{aligned} v_{s_{1,\perp}} &= (L_{44}/\rho)^{1/2} = V_{s,\parallel} \\ v_{s_{2,\perp}} &= (L_{66}/\rho)^{1/2}. \end{aligned} \tag{4.25}$$

The effective elastic wave velocities for PTI and CTI bodies are plotted in Figs. 5 and 6 for various ω and for $\nu = 1/4$. The dotted curves in the graphs of v_p are the predictions of Griggs *et al.*[5], made on the assumption that there is a dilute concentration of dry circular cracks in the body. For small values of crack density, this prediction is close to that of the self-consistent approximation, but they soon diverge rather substantially at rather modest crack densities.

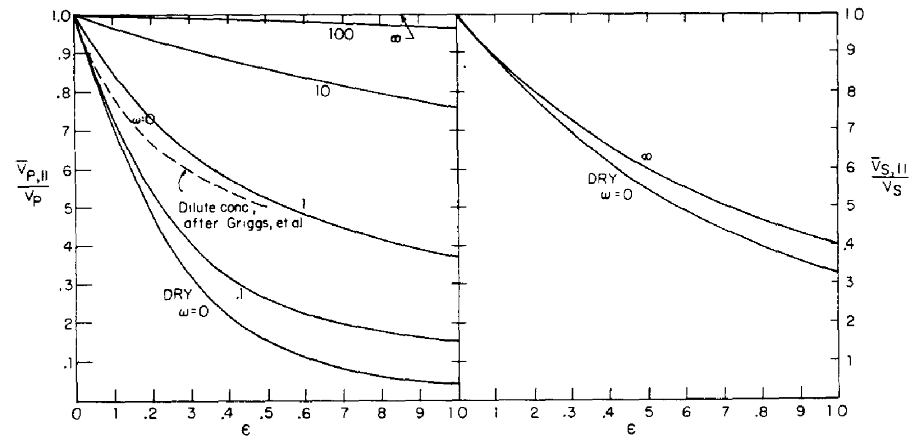

Fig. 5(a). PTI seismic velocities (circular cracks). Propagation in direction of unique axis.

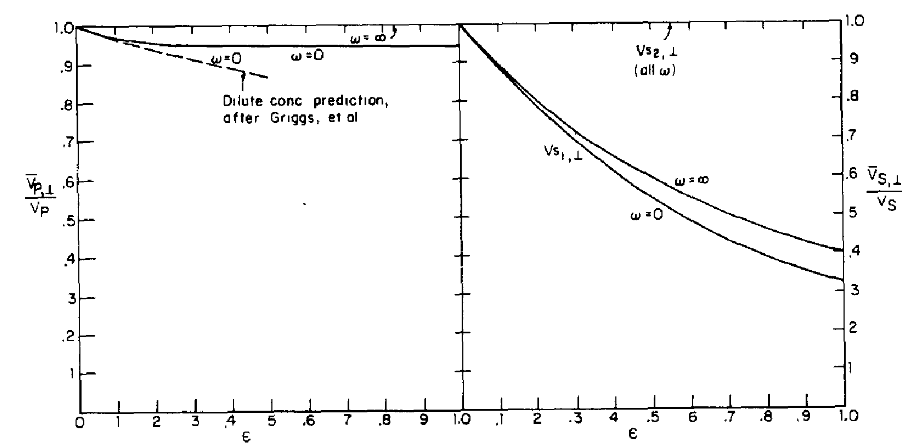

Fig. 5(b). PTI seismic velocities (circular cracks). Propagation in plane of isotropy.

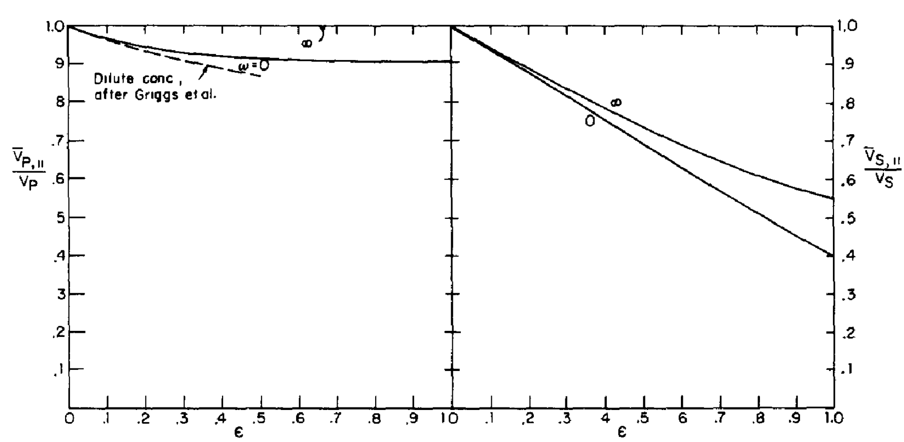

Fig. 6(a). CTI seismic velocities (circular cracks). Propagation in direction of unique axis.

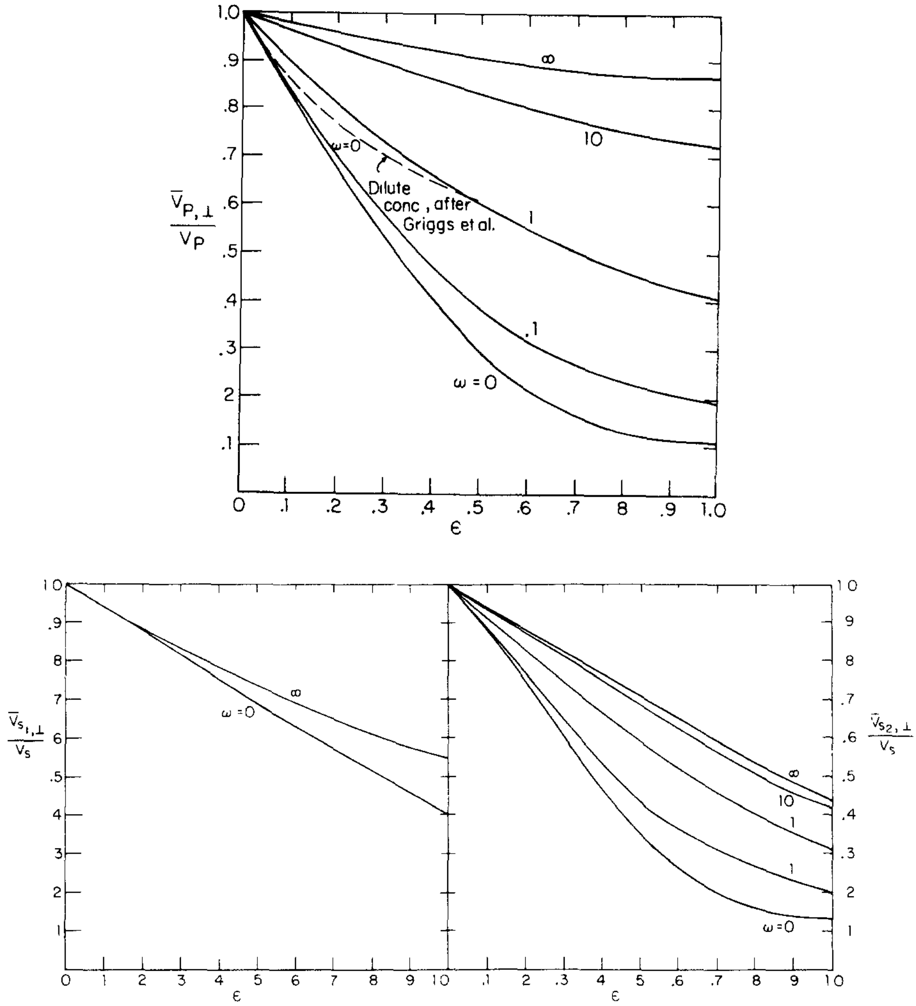

Fig. 6(b). CTI seismic velocities (circular cracks). Propagation in plane of isotropy, $\bar{V}p, \perp$, $\bar{V}s_1, \perp$, $\bar{V}s_2, \perp$.

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